On Complete Continuity of P. S. Uryson's Operator in Function Spaces

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Abstract

The purpose of this paper is to give conditions of both the continuity and compactness of Uryson's operator $\int K[s, t, \phi(t)]\, dt$ which acts in modulared function spaces.

1. Introduction. In non-linear integral equations, the complete continuity of an operator from which the equation is produced plays a very impotant role, for example, the existence of solutions or eigen-functions in the equations. (cf. M. A. Krasnosel'skii[3] and S. Yamamuro[13])

A sufficient condition of the complete continuity of Uryson's operator acting in the space C, as the totality of all continuous functions on a compact subset in Euclidean space, have been given by L. A. Ladyzhenskii[4].

In case the operator acts from the space L_{p_1} $(p_1>1)$ to the space L_{p_2} $(p_2>1)$, M. A. Krasnosel'skii and L. A. Ladyzhenskii have given some sufficient conditions of the complete continuity, but it seems that one result has a defect, so far as we see the fact described in [Amer. Math. Soc. Transl. Ser. 2, vol. 10, p 352].

In this paper, we will consider the operator acting in modulared function spaces with some restrictions, which was defined by H. Nakano[7], and we give some sufficient conditions for the complete continuity of the operator. (see Theorem 4 and 5)

2. Preliminaries. In this section, we will state an outline of modulared function spaces and fundamental definitions.

Let Δ be a bounded subset in Euclidean space and mes $(\Delta)=1$.

let $\Phi(\xi, x)$ $(\xi\geqq 0, x\in\Delta)$ be measurable on Δ for each $\xi\geqq 0$ and non-decreasing convex function of $\xi\geqq 0$ for which satisfies:

1) $\Phi(0, x)=0$ for all $x\in\Delta$;
2) $\lim_{\xi\to\alpha-0} \Phi(\xi, x)=\Phi(\alpha, x)$ for each $x\in\Delta$;
3) $\lim_{\xi\to+\infty} \Phi(\xi, x)=+\infty$ for each $x\in\Delta$;
4) for any $x\in\Delta$, there exists $\alpha=\alpha(x)>0$ such that $\Phi(\alpha, x)>+\infty$.

The modulared function space $\mathbf{L}_\Phi(\Delta)$ is a totality of all measurable functions $\phi(x)$ on Δ such that

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$$m(\alpha\phi)=\int_{\varDelta}\varPhi(\alpha|\phi(x)|,x)\,dx<+\infty \qquad \text{for some } \alpha>0.$$

When we define a **semi-order** (or partial order) in $\mathbf{L}_\varPhi$ by the relation that $\phi\geqq\psi$ if and only if $\phi(x)\geqq\psi(x)$ except for a set of measure zero, the space $\mathbf{L}_\varPhi$ is a supperuniversally continuous semiordered linear space*.

The above functional $m(\phi)$ on $\mathbf{L}_\varPhi$ is called a modular on $\mathbf{L}_\varPhi$ and satisfies the modular conditions[7]:

1) $0\leqq m(\phi)\leqq+\infty$ for all $\phi\in\mathbf{L}_\varPhi$;
2) if $m(\xi\phi)=0$ for all $\xi\geqq 0$, then $\phi=0$
3) for any $\phi\in\mathbf{L}_\varPhi$ there exists $\alpha>0$ such that $m(\alpha\phi)<+\infty$;
4) for every $\phi\in L_\varPhi$, $m(\xi\phi)$ is a convex function of $\xi\geqq 0$;
5) $|\phi|\leqq|\psi|$ implies $m(\phi)\leqq m(\psi)$;
6) $|\phi|\cap|\psi|=0$ implies $m(\phi)+\psi)=m(\phi)+m(\psi)$;
7) $0\leqq\phi_\lambda\uparrow_{\lambda\in\varDelta}\phi$** implies $m(\phi)=\sup_{\lambda\in\varDelta} m(\phi_\lambda)$.

Writing the left-derivative of $\varPhi(\xi,x)$ at ξ by $\varphi(\xi,x)$ with $\varphi(0,x)\equiv 0$, we have a measurable in x and non-decreasing function $\varphi(\xi,x)$ in $\xi\geqq 0$. If we define an inverse function $\psi(\eta,x)$ of $\varphi(\xi,x)$ as $\eta=\varphi(\xi,x)$, such that it is non-decreasing function of $\eta\geqq 0$, $\psi(0,x)\equiv 0$ and

$$\psi(\eta-0,x)\leqq\xi\leqq\psi(\eta+0,x) \qquad \text{for } \eta=\varphi(\xi,x),$$
$$\varphi(\xi-0,x)\leqq\eta\leqq\varphi(\xi+0,x) \qquad \text{for } \xi=\psi(\eta,x),$$

then the function:

$$\varPsi(\eta,x)=\int_0^\eta\psi(\eta,x)\,d\eta$$

is measurable in $x\in\varDelta$ and satisfies the same conditions as $\varPhi(\xi,x)$. Furtheremore, we have Young's inequality

$$\xi\eta\leqq\varPhi(\xi,x)+\varPsi(\eta,x)$$

for $\xi,\eta\geqq 0$ and $x\in\varDelta$, with equality if one at least of the relations

$$\varphi(\xi-0,x)\leqq\eta\leqq\varphi(\xi+0,x),\quad \psi(\eta-0,x)\leqq\xi\leqq\psi(\eta+0,x)$$

is satisfied. By the function $\varPsi(\eta,x)$, the space $\mathbf{L}_\varPsi$ which is called a **conjugate space** of $\mathbf{L}_\varPsi$ is defined, and further a modular $\overline{m}(\psi)$ on $\mathbf{L}_\varPsi$, i. e.

$$\overline{m}(\psi)=\int_{\varDelta}\varPsi(|\psi(x)|,x)\,dx$$

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* A semi-ordered linear space R is said to be supperuniversally continuous, if for any system $a_\lambda\geqq 0$ ($\lambda\in\varDelta$) there exist countable a_{λ_ν} ($\lambda\in\varDelta$) and $a\in R$ for which $a=\bigcap_{\nu=1}^{\infty}a_{\lambda_\nu}=\bigcap_{\lambda\in\varDelta}a_\nu$, where $\cap\, a_\lambda$ means a infimum of a_λ.

** $\phi_\lambda\uparrow_{\lambda\in\varDelta}\phi$ means that for any $\lambda,\mu\in\varDelta$ teere exists $\rho\in\varDelta$ such that $\phi_\lambda\cup\phi_\mu\leqq\phi_\rho$, and $\bigcap_{\mu\in\varDelta}\bigcup_{\lambda\geqq\mu}\phi_\lambda=\bigcup_{\mu\in\varDelta}\bigcap_{\lambda\geqq\mu}\phi_\lambda=\phi$, where $\bigcup_{\lambda\in\varDelta}\phi_\lambda$ is a supremum of ϕ_λ.

is defined as follows:

$$\overline{m}(\psi)=\sup\left\{\int_{\varDelta}\psi(x)\,\phi(x)\,dx-m(\phi)\right\}$$

where $\overline{m}$ is called a **conjugate modular** of m.

In the space $\mathbf{L}_{\varPhi}$, defining two kinds of norms:

$$|||\phi|||_{\varPhi}=\inf_{m(\xi\phi)\leqq 1}\frac{1}{|\xi|}\,;\quad \|\phi\|_{\varPhi}=\inf_{\xi>0}\frac{1+m(\xi\phi)}{\xi},$$

we have $|||\phi|||_{\varPhi}\leqq\|\phi\|_{\varPhi}\leqq 2\,|||\phi|||_{\varPhi}$, and their norms are both monotone complete norms*, so $\mathbf{L}_{\varPhi}$ is Banach space, because the above modular on $\mathbf{L}_{\varPhi}$ is monotone complete*.

As examples of such spaces, we can denote the well-known following spaces.

Orlicz space[5, 15)]**, i. e., for a non-decreasing left-continuous function $\varphi(\xi)$ on $[0,\infty)$ with $\varphi(0)=0$, putting

$$\varPhi(u)=\int_0^u\varphi(\xi)\,d\xi \qquad (u\geqq 0)$$

the totality of all measurable functions $\phi(x)$ on $\varDelta$ such that

$$\int_{\varDelta}\varPhi(\alpha|\phi(x)|\,dx<+\infty \qquad \text{for some } \alpha>0.$$

Space $\mathbf{L}_{p(x)}$, i. e., for a measurable function $p(x)\geqq 1\ (x\in\varDelta)$, the totality of all measurable functions $\phi(x)$ on $\varDelta$ such that

$$\int_{\varDelta}\frac{1}{p(x)}|\alpha\phi(x)|^{p(x)}\,dx<+\infty \qquad \text{for some } \alpha>0.$$

A modular $m(\phi)$ on $\mathbf{L}_{\varPhi}$ is said to be upper bounded modular, if there exist $\alpha,\ \gamma>1$ such that

$$\varPhi(\alpha\xi,x)\leqq\gamma\varPhi(\xi,x) \qquad \text{for all } \xi\geqq 0,\ x\in\varDelta,$$

And, m is said to be lower bounded modular, if there exist $\alpha>\gamma>1$ such that

$$\varPhi(\alpha\xi,x)\geqq\gamma\varPhi(\xi,x) \qquad \text{for all } \xi\geqq 0,\ x\in\varDelta.$$

If m is lower (upper) bounded then its conjugate modular $\overline{m}$ is upper (lower) bounded.

m is said to be bounded modular if it is upper and lower bounded modular. If m is a bounded modular, then $\mathbf{L}_{\varPhi}$ is reflexive as Banach space with the above norms[2)], for instance, $\mathbf{L}_p\ (p>1)$ and Orlicz spaces defining by complementary Young's functions $\varPhi(u)$ and $\varPsi(v)$ for which satisfy both $(\varDelta^2)$-condition.

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* A norm $\|\phi\|$ is called to be monotone complete if $0\leqq\phi_\nu\uparrow_{\nu=1}^{\infty}$ and $\sup_{\nu\geqq 1}\|\phi_\nu\|<+\infty$ implies the existence of an element ϕ such that $\phi_\nu\uparrow_{\nu=1}^{\infty}\phi$. A monotone completeness of a modular implies a monotone completeness of a norm, and a monotone completeness of a norm implies a completeness in ussal sense.[1, 14)]

** Orlicz spaces are modulared function spaces with constant modulars[7)].

Throughout this paper we assume that the modulared function spaces $\mathbf{L}_{\Phi_i}$ and their conjugate spaces $\mathbf{L}_{\Psi_i}$ $(i=1, 2)$ have the bounded modulars, and the functions $\varphi_i(1, x)$, $\psi_i(1, x)$ are integrable on $\varDelta$, where φ_i and ψ_i are the left-derivatives of $\boldsymbol{\Phi}_i$ and Ψ_i respectively.

The integral operator:

$$\mathbf{A}\phi(s) = \int_{\varDelta} K[s, t, \phi(t)]\, dt$$

is called the operator of P. S. Uryson, where the function $K[s, t, u]$ is defined for $(s, t) \in \varDelta \times \varDelta$ and for real number u.

In this paper, we will deal with the case which $K[s, t, u]$ is continuous in u for fixed (s, t) and measurable in the remainder of the variables for fixed u.

A subset F of Banach space E is called to be **compact** (**weakly compact**), if every infinite subset contains a subsequence converging (weak converging) in E.

An operator is called to be **bounded** if it transforms every bounded (in the norm) subset of Banach space E_1 into a set which is bounded (in the norm) in Banach space E_2.

An operator $\mathbf{A}$, acting from E_1 into E_2, is called to be **continuous at the point** $\phi_0 \in E_1$ if, for every sequence $\{\phi_n\}$ converging to ϕ_0, $\{\mathbf{A}\phi_n\}$ converges to $\mathbf{A}\phi_0$ in E_2. An operator is called to be **continuous** on E if it is continuous at each point of E.

An operator $\mathbf{A}$ is called to be **compact** if it transforms every bounded set into a compact set.

An operator $\mathbf{A}$ is called to be **completely continuous** if it is continuous and compact.

3. In this section, we will consider a sufficient condition of both the boundedness and continuity of Uryson's operator which acts from the space $\mathbf{L}_{\Phi_1}$ with a modular m_1 into the space $\mathbf{L}_{\Phi_2}$ with a modular m_2.

Lemma 1. *If $K[s, t, u]$ $(s, t \in \varDelta,\ -\infty < u < +\infty)$ is measurable on $\varDelta \times \varDelta$ for fixed u and continuous in u for fixed (s, t), then for any $a \leqq b$ there exists a bounded measurable function $h(s, t)$ on $\varDelta \times \varDelta$ such that*

$$\sup_{a \leqq u \leqq b} |K[s, t, u]| = |K[s, t, h(s, t)]| \qquad \textit{for each } s \textit{ and } t.$$

Proof. First, we shall show the measurability of the function

$$k(s, t) = \sup_{a \leqq u \leqq b} |K[s, t, u]|.$$

When we put, for any positive number α,

$$E_\alpha = \{(s, t);\ k(s, t) \leqq \alpha,\ F^r_{\alpha, n} = \{(s, t);\ |K[s, t, u_r]| < \alpha + 1/n\}$$

and
$$E_\alpha = \bigcap_{n=1}^{\infty} \bigcap_{u_r} F^r_{\alpha, n}{}^*,$$

* $\cap$ means the intersection of sets.

where $\{u_r\}$ is a totality of all rational numbers in the closed interval $[a, b]$ and n is a natural number, we get a measurability of subset F_α of $\Delta\times\Delta$. Furthermore, we can see easily an equality $E_\alpha=F_\alpha$ so that E_α is a measurable subset of $\Delta\times\Delta$. Thus $k(s, t)$ is measurable on $\Delta\times\Delta$.

Next, we define the function $h(s, t)$ as, for each (s, t), a maximum value of u's for which hold the relations $k(s, t)=|K[s, t, u]|$.

For any β $(a\leqq\beta\leqq b)$, putting

$$E_\beta=\{(s, t)\,;\ h(s, t)\leqq\beta\}$$

$$F_\alpha^n=\{(s, t)\,;\ \sup_{a\leqq\beta+1/n\leqq b}|K[s, t, u]|<\sup_{a\leqq u\leqq\beta}|K[s, t, u]|\}$$

and
$$F_\beta=\bigcap_{a\leqq\beta+1/n\leqq b}F_\beta^n$$

where n is a natural number, we have also a measurable subset F_β of $\Delta\times\Delta$ and an equality $E_\beta=F_\beta$, and hence $h(s, t)$ is measurable on $\Delta\times\Delta$. It is obvious that $h(s, t)$ is bounded on $\Delta\times\Delta$. We state the following:

Theorem 1. *Let $K[s, t, u]$ $(s, t\in\Delta,\ -\infty<u<+\infty)$ be continuous in u for fixed s and t, and measurable on $\Delta\times\Delta$ for fixed u.*

If it satisfies the following conditions:

a) *for every bounded measurable function $h(s, t)$ on $\Delta\times\Delta$*

$$m_2\left(\int_\Delta K[s, t,\ h(s, t)\ dt\right)<+\infty\,;$$

b) *for any $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that $\|\phi-\psi\|_{\Phi_1}<\delta$ implies*

$$m_2\left(\int_F\{K[s, t, \phi]-K[s, t, \psi]\}\ dt\right)<\varepsilon$$

for mes $(F)<\delta\,(F\subset\Delta)$, *then Uryson's operator $\mathbf{A}\phi$ acts from $\mathbf{L}_{\Phi_1}$ into $\mathbf{L}_{\Phi_2}$, and is bounded and continuous.*

Proof. We prove at first that $\mathbf{A}\phi$ acts from $\mathbf{L}_{\Phi_1}$ into $\mathbf{L}_{\Phi_2}$ and is bounded. For any $\phi(t)\in\mathbf{L}_{\Phi_1}$, taking $\varepsilon=1$ in b) there exists $\delta=\delta(1)>0$ such that

$$m_2\left(\int_F\{K[s, t, \phi]-K[s, t, \psi]\}\ dt\right)<1$$

for $\|\phi-\psi\|_{\Phi_1}<\delta$ and mes$(F)<\delta$. Since we can select $\phi_i\in\mathbf{L}_{\Phi_1}$ $(i=0, 1, \cdots, k)$ such that $\phi_0=\phi$, $\|\phi_i-\phi_{i-1}\|_{\Phi_1}<\delta\,(i=1, 2, \cdots, k)$ and $\phi_k=0$, where $k=[\|\phi\|_{\Phi_1}/\delta]^*+1$, we have, by the convexity of Φ_2,

$$m_2\left(\frac{1}{k+1}\int_E K[s, t, \phi]\ dt\right)$$
$$\leqq\sum_{i=0}^{k-1}\frac{1}{k+1}m_2\left(\int_F\{K[s, t, \phi_i]-K[s, t, \phi_{i+1}]\}\ dt\right)$$
$$+\frac{1}{k+1}m_2\left(\int_F K[s, t, 0]\ dt\right)\leqq\frac{k+A}{k+1}<1+A$$

* $[x]$ is the symbol of Gauss.

where $A=m_2\left(\int_\Delta |K[s,t,0]|\,dt\right)$, and

$$m_2\left(\int_F K[s,t,\phi]\,dt\right)\leqq B\cdot m_2\left(\frac{1}{k+1}\int_F K[s,t,\phi]\,dt\right)$$

where B is only dependent on k, because m_2 is the upper bounded modular. Therefore, for a patition $\{\Delta_1, \Delta_2, \cdots, \Delta_j\}$ of Δ, which satify mes $(\Delta_i)<\delta$ for $i=1, 2, \cdots, j$ where $j=[1/\delta]+1$, we have

$$\begin{aligned} m_2\left(\frac{1}{j}\int_F K[s,t,\phi]\,dt\right) &\leqq \sum_{i=1}^{j}\frac{1}{j}\,m_2\left(\int_{\Delta_i} K[s,t,\phi]\,dt\right) \\ &\leqq B(1+A), \end{aligned}$$

and hence it follows that, by the upper boundedness of m_2,

$$\begin{aligned} m_2\left(\int_\Delta K[s,t,\phi]\,dt\right) &\leqq C\cdot m_2\left(\frac{1}{j}\int_\Delta K[s,t,\phi]\,dt\right) \\ &\leqq CB(1+A) \end{aligned}$$

where C is only dependent on j. Thus, it is shown that $\mathbf{A}\phi(s)\in \mathbf{L}_{\Phi_2}$ and further $\|\phi\|_{\Phi_1}\leqq \gamma$ implies $m_2(\mathbf{A}\phi) < CB(1+A)$, that is, $\|\mathbf{A}\phi\|_{\Phi_2}<2\,CB(1+A)$*, where $k=[\gamma/\delta]+1$.

Next, we prove the continuity of the operator $\mathbf{A}\phi$.

If $\lim_{k\to\infty}\|\phi_k-\phi_0\|_{\Phi_1}=0$ (ϕ_k, $\phi_0\in\mathbf{L}_{\Phi_1}$) then $\{|\phi_k-\phi_0|\}$ converge to 0 weakly and hence $\lim_{k\to\infty}\int_\Delta|\phi_k(t)-\phi_0(t)|\,dt=0$. Accordingly, we can select a subsequence $\{\phi_n(t)\}$ converging to $\phi_0(t)$ for almost all t.

Since $\phi_0(t)$ is almost all finite on Δ, for any natural number ν there exist $M_\nu>0$ and a subset $\mathrm{E}_\nu\subset\Delta$ such that mes $(\mathrm{E}_\nu)\geqq 1-1/\nu$ and $|\phi_0(t)|\leqq M_\nu$ for all $t\in\mathrm{E}_\nu$.

Furthermore, by Egoroff's theorem, for any $\varepsilon>0$ there exists a subset $\mathrm{X}\subset\Delta$ such that

mes $(\Delta-\mathrm{X})<\varepsilon$ and $\{\phi_n\}$ converge to ϕ_0 uniformly on X.

Putting $\mathrm{X}_\nu=\mathrm{X}\cap\mathrm{E}_\nu$, we have mes $(\Delta-\mathrm{X}_\nu)\leqq\varepsilon+1/\nu$ and for all of sufficiently large n,

$$\left|\int_{X_\nu} K[s,t,\phi_n(t)]\,dt\right| \leqq \int_{X_\nu}\sup_{|M_\nu-u|\leqq\varepsilon}|K[s,t,u]|\,dt<+\infty$$

for almost all s, beacause, by Lemma 1, the assumption a) implies

$$m_2\left(\int_\Delta \sup_{|M_\nu-u|\leqq\varepsilon}|K[s,t,u]|\,dt\right)<+\infty,$$

and hence

* This is obtained from the fact that $m_2(x)\leqq 1$ implies $\|x\|_{\Phi_2}\leqq 1$.

** The step function $f(t)=1$ on Δ belongs to the conjugate space $\mathbf{L}_{\Psi_1}$ of $\mathbf{L}_{\Phi_1}$, because

$$\int_\Delta \Phi_1(\psi_1(1,x),x)\,dx+\int_\Delta \Phi_1(1,x)\,dx=\int_\Delta \psi_1(1,x)\,dx<+\infty.$$

$$\int_{\varDelta} \sup_{|M_\nu - u| \leqq \varepsilon} |K[s, t, u]| \, dt \in \mathbf{L}_{\Phi_2}.$$

Therefore, by Lebesgue's theorem, we have

$$\lim_{n\to\infty} \int_{X_\nu} K[s, t, \phi_n] \, dt = \int_{X_\nu} K[s, t, \phi_0] \, dt$$

for all ν and almost all s, since $K[s, t, \phi_n]$ converge to $K[s, t, \phi_0]$ for almost all t, s, and consequently it follows that

$$\lim_{n\to\infty} \Phi_2\left(\left|\int_{X_\nu}\left\{K[s, t, \phi_n] - K[s, t, \phi_0]\right\} dt, s\right) = 0$$

for almost all s.

And, we have for all of sufficiently large n

$$\begin{aligned} &\Phi_2\left(\frac{1}{2}\left|\int_{X_\nu}\left\{K[s, t, \phi_n] - K[s, t, \phi_0]\right\} dt\right|, s\right) \\ &\leqq \frac{1}{2}\Phi_2\left(\left|\int_{X_\nu} K[s, t, \phi_n]\, dt\right|, s\right) + \frac{1}{2}\Phi_2\left(\left|\int_{X_\nu} K[s, t, \phi_0]\, dt\right|, s\right) \\ &\leqq \Phi_2\left(\int_{X_\nu} \sup_{|M_\nu - u| \leqq \varepsilon} |K[s, t, u]| \, dt, s,\right), \end{aligned}$$

and the last term is integrable by a) and Lemma 1, so that, by Lebesgue's theorem,

$$\lim_{n\to\infty} \int_{\varDelta} \Phi_2\left(\left|\int_{X_\nu}\left\{K[s, t, \phi_n] - K[s, t, \phi_0]\right\} dt\right|, s\right) ds = 0,$$

because m_2 is upper bounded.

Now, for any $\varepsilon > 0$, when we select ν, ε_1 in the above as which satisfy $\varepsilon_1 + 1/\nu < \delta$ where $\delta = \delta(\varepsilon)$ is the number in the assumption b), there exists $n_0 = n_0(\varepsilon)$ such that

$$\|\phi_n - \phi_0\|_{\Phi_1} < \delta \text{ and } \int_{\varDelta} \Phi_2\left(\left|\int_{X_\nu}\left\{K[s, t, \phi_n] - K[s, t, \phi_0]\right\} dt\right|, s\right) ds < \varepsilon,$$

and consequently, it follows that, by the convexity and upper boundedness of m_2,

$$m_2(\mathbf{A}\phi_n(s) - \mathbf{A}\phi_0(s)) < N \cdot \varepsilon$$

where N is a constant for which satisfies

$$\Phi_2(2\xi, s) \leqq N \cdot \Phi_2(\xi, s) \qquad \text{for all } \xi \geqq 0 \text{ and } s.$$

This shows that $\{\mathbf{A}\phi_n\}$ converges to $\mathbf{A}\phi_0$ by the modular* and hence it follows that

$$\lim_{n\to\infty} \|\mathbf{A}\phi_n - \mathbf{A}\phi_0\|_{\Phi_2} = 0.$$

If we suppose that $\lim_{k\to\infty} \|\phi_k - \phi_0\|_{\Phi_1} = 0$ and

* If a modular m is upper bounded, then $\lim_{n\to\infty} m(\xi(x_n - x)) = 0$ for all $\xi \geqq 0$ is equivalent to $\lim_{n\to\infty} m(x_n - x) = 0$, and that the modular convergence coincides with the norm convergence. (cf. H. Nakano[7])

$(\sharp)\qquad \|\mathbf{A}\phi_k-\mathbf{A}\phi_0\|_{\Phi_2}\geqq\varepsilon\qquad$ for some $\varepsilon>0$ and $k=1,2,\cdots,$

then we can find a subsequence $\{\phi_n(t)\}$ converging to $\phi_0(t)$ in almost all $t\in\Delta$ and hence it follows, as is shown above, that

$$\lim_{n\to\infty}\|\mathbf{A}\phi_n-\mathbf{A}\phi_0\|_{\Phi_2}=0.$$

This is contradiction to $(\sharp)$. Thus the operator is continuous.

Remark. In the operator of Hammerstein type, i.e.

$$\mathbf{H}\phi(s)=\int_\Delta K(s,t)f(t,\phi(t))\,dt$$

it is known that the operator $\mathbf{H}\phi$ is continuous (moreover, it is compact) in Orlicz space $\mathbf{L}_\Phi^*$* if it satisfies the following conditions:

1) $$\int_\Delta \Phi\left(\int_\Delta \Psi_1\left(\left|K(s,t)\right|\right)dt\right)ds>+\infty\,;$$

2) $$|f(t,u)|\leqq a(t)+\Phi^{-1}(b\Phi(|u|))$$

where $a(t)\in\mathbf{L}_{\Phi_1}^*$, $b>0$ and Φ, Φ_1 and their complementary Young's functions Ψ, Ψ_1 satisfy the (Δ_2)-condition.[3,11,12,15]

Those conditions satisfy the condtions in Theorem 1, because the condition 2) implies the boundedness of the operator $\mathbf{f}$:

$$\mathbf{L}_\Phi^*\ni\phi(t)\to f(t,\phi(t)\in\mathbf{L}_{\Phi_1}^*,$$

and also the bounded set $\mathfrak{A}$ in $\mathbf{L}_{\Phi_1}^*$ is the absolutely equi-continuous integrals[8], since

$$\int_\Delta f(x)\cdot\phi_1(f(x))\,dx\leqq M<+\infty\qquad\text{for all } f(x)\in\mathfrak{A},$$

where ϕ_1 is a left-derivative of Ψ_1, consequently, the condition b) is satisfied.

4. In this section, we will consider the compactness of Uryson's operator. L. A. Ladyzhenskii[7] given a sufficient condition of the compactness of the operator acting in the space $\mathbf{C}$, which it is proved by use of Ascoli-Arzela's theorem. V. V. Nemyckii[9] shown a sufficient condition of the compactness of the operator in the space $\mathbf{C}$ and his proof is placed on the basis of Kolmogoroff's criteria concerning for a compactness of a set. Those conditions have been established under the assumption that Δ is bounded closed set in n-dimensional Euclidian space $\mathbf{R}_n$ with Lebesgue measure.

We will give a theorem concerning for the compactness of the operator which acts in modlared function spaces defining on a bounded set in $\mathbf{R}_n$.

Throem 2. *Let the operator* $\mathbf{A}\phi$ *be the bounded operator which acts from the unit sphere* $\mathbf{S}_1$ *of* $\mathbf{L}_{\Phi_1}$ *into* $\mathbf{L}_{\Phi_2}$. *Further, if it satisfies the condition*

$$(\sharp\sharp)\qquad \int_\Delta |K[x,t,\phi(t)]-K[s,t,\phi(t)]|\,dt\leqq f(s)\cdot p(h)\quad(\phi\in\mathbf{S}_1)$$

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* L_Φ^* means the Orlicz space satisfying (Δ_2)-condition. cf. A. C. Zaanen[15]

for $\|x-s\|\leqq h$ ($\|x\|$ *is the usual norm in* $\mathbf{R}_n$), *where* $f(s)\in \mathbf{L}_{\Phi_2}$ *and* $p(h)$ *is some real function tending to zero as* $h\to 0$, *then* $\mathbf{A}\phi$ *is the compact operator from* $\mathbf{S}_1$ *into* $\mathbf{L}_{\Phi_2}$.

Proof. Putting

$$(\mathbf{A}\phi(s))^\delta=\frac{1}{V(\delta)}\int_{U(s,\delta)}\mathbf{A}\phi(x)\,dx$$

where $V(\delta)$ is the volume of $U(s,\delta)$ which is the sphere with the center s and the radius δ, we have, by (♯♯),

$$\begin{aligned}&\Phi_2(|\mathbf{A}\phi-(\mathbf{A}\phi)^\delta|,\ s)\\ &\leqq \Phi_2\Big(\frac{1}{V(\delta)}\int_U\Big|\int_\Delta\Big\{K[s,t,\phi]-K[x,t,\phi]\Big\}\,dt\,\Big|\,dx,\ s\Big)\\ &\leqq \Phi_2(f(s)\,p(\delta),\ s)\ \text{ for almost all } s\in\Delta \text{ and all } \phi\in\mathbf{S}_1,\end{aligned}$$

and the last term is integrable on Δ.

On the other hand, we have obviously

$$\lim_{\delta\to 0}\Phi_2(f(s)\,p(\delta),s)=0 \qquad \text{for almost all } s\in\Delta.$$

Therefore, we have

$$\lim_{\delta\to 0} m_2(\mathbf{A}\phi-(\mathbf{A}\phi)^\delta)=0 \qquad \text{uniformly on } \mathbf{S}_1,$$

i. e., for any $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that

$$\|\mathbf{A}\phi-(\mathbf{A}\phi)^\delta\|_{\Phi_2}<\varepsilon \qquad \text{for all } \phi\in\mathbf{S}_1.$$

Accordingly, if it is shown that $\{(\mathbf{A}\phi)^\delta\}$ ($\phi\in\mathbf{S}_1$) is compact set in $\mathbf{L}_{\Phi_2}$ then the compactness of the operator $\mathbf{A}\phi$ is obvious.

Since $\mathbf{L}_{\Phi_2}$ is reflexive as Banach space, the boundedness of $\{\mathbf{A}\phi\}$ ($\phi\in\mathbf{S}_1$) implies the weak compactness of $\{\mathbf{A}\phi\}$ ($\phi\in\mathbf{S}_1$). Therefore, for any infinite sequence in $\{\mathbf{A}\phi\}$ ($\phi\in\mathbf{S}_1$) we can find a subsequence, such that for every $\delta>0$

$$\lim_{n\to\infty}(\mathbf{A}\phi_n(s))^\delta=(\psi_0(s))^{\delta}* \qquad \text{for almost all } s\in\Delta,$$

where $\psi_0(s)$, $(\psi_0(s))^\delta\in\mathbf{L}_{\Phi_2}$**. Also, we have

$$\begin{aligned}|(\mathbf{A}\phi_n(s))^\delta-(\psi_0(s))^\delta| &\leqq \frac{1}{V(\delta)}|||\chi_\Delta|||_{\Psi_2}\cdot\|\mathbf{A}\phi_n-\psi_0\|_{\Phi_2}\\ &\leqq \frac{1}{V(\delta)}|||\chi_\Delta|||_{\Psi_2}\,\{M+\|\psi_0\|_{\Phi_2}\}\end{aligned}$$

for almost all $s\in\Delta$, where $M=\sup_{\phi\in\mathbf{S}_1}\|\mathbf{A}\phi\|_{\Phi_2}$.

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* Weakly convergent sequence $\{A\phi_n\}$ is the requirement, because all step functions belong to L_{Ψ_2}.

** It follows that $\left|\int_{U(s,\delta)}\psi_0(x)\,dx\right|\leqq|||\chi_\Delta|||_{\Psi_2}\|\psi_0\|_{\Phi_2}$ and all step functions on Δ belong to the spaces L_{Φ_2} and L_{Ψ_2}.

Therefore, we have

$$\lim_{n\to\infty} \|(\mathbf{A}\phi_n(s))^\delta-(\phi_0(s))^\delta\|_{\Phi_2}=0,$$

namely, $\{(\mathbf{A}\phi)^\delta\}$ $(\phi\in\mathbf{S}_1)$ is compact in $\mathbf{L}_{\Phi_2}$ by the definition.

Theorem 3. *When $\mathbf{L}_{\Phi_i}$ $(i=1,2)$ are Orlicz spaces satisfying the (Δ_2)-condition, we can replace the condition (##) by the waeker conditions: for almost all s*

$$(\sharp\,\sharp\sharp)\qquad \lim_{\substack{x\to s\\ x,s\in\Delta}} \int_\Delta |K[x,t,\phi]-K[s,t,\phi]|\,dt=0 \qquad \text{uniformly on } \mathbf{S}_1;$$

$$(\sharp\sharp\,\sharp\sharp)\qquad \sup_{\phi\in\mathbf{S}_1} |\mathbf{A}\phi(s)|=f(s)\in\mathbf{L}_{\Phi_2}.$$

Proof. Since we know easily

$$\begin{aligned}\Phi_2(|\mathbf{A}\phi-(\mathbf{A}\phi)^\delta|)&\leqq C\left\{\Phi_2(|\mathbf{A}\phi|+\Phi_2\Big(\frac{1}{V(\delta)}\Big|\int_\Delta\int_{U(s,\delta)}K[x,t,\phi]\,dx\,dt\Big|\Big)\right\}\\ &\leqq C\left\{\Phi_2(|f(s)|)+\Phi_2\Big(\frac{1}{V(\delta)}\|\chi_U\|_{\Phi_2}\cdot\|\mathbf{A}\phi\|_{\Phi_2}\Big)\right\}\\ &\leqq C\left\{\Phi_2(|f(s)|+\Phi_2(\alpha\cdot M)\right\}\in\mathbf{L}_{\Phi_2}\end{aligned}$$

where $\alpha=\lim_{\xi\to\infty}\Psi_2^{-1}(\xi)/\xi<+\infty$*, $M=\sup_{\phi\in\mathbf{S}_1}\|\mathbf{A}\phi\|_{\Phi_2}$ and C is some constant, the theorem is proved by the same method as the proof of Theorem 2.

Lemma 2. *If $\mathbf{L}_\Phi(\Delta,\mu)$ is a modulared function space, defining on a bounded set Δ in $\mathbf{R}_n$, with the upper bounded modular, then for any $\varphi\in\mathbf{L}_\Phi$, we have*

$$\lim_{\|h\|\to\phi}\|\varphi(x+h)-\varphi(x)\|_\Phi=0$$

where $\varphi(x+h)=0$ if $x+h\in\Delta$ and $\|h\|$ is the usual norm in $\mathbf{R}_n$.

Proof. For any $\varepsilon>0$ and $\varphi\in\mathbf{L}_\Phi$, there exists a closed subset G of Δ such that $\|\varphi-\varphi_G\|<\varepsilon$ where

$$\varphi_G(t)=\begin{cases}\varphi(t) & \text{if } t\in G\\ 0 & \text{if } t\notin G\end{cases}$$

and $\varphi_G\in\mathbf{L}_\Phi$.

Therefore, we will prove the lemma for a function on G.

(i) Putting, for $x\in G$

$$\varphi_n(x)=\begin{cases} n & \text{if } \varphi(x)\geqq n\\ \varphi(x) & \text{if } -n\leqq\varphi(x)\leqq n\\ -n & \text{fi } \varphi(x)\leqq -n,\end{cases}$$

we have $\lim_{n\to\infty}|\varphi_n(x)-\varphi(x)|=0$ for almost all $x\in G$ and $|\varphi_n(x)-\varphi(x)|\leqq 2|\varphi(x)|$,

* $\lim_{\xi\to\infty}\Psi_2^{-1}(\xi)/\xi<+\infty$ is equivalent to $\lim_{\eta\to\infty}\eta/\Psi_2(\eta)=0$ and $\|\chi_U\|_{\Phi_2}\leqq 1/\Psi_2(1/V(\delta))$.

therefore, it follows that
$\lim_{n\to\infty} \|\varphi_n-\varphi\|_\Phi=0$, i. e., for any $\varepsilon>0$, there exists $n_0=n_0(\varepsilon)>0$ such that

$$\|\varphi_{n_0}-\varphi\|_\Phi<\varepsilon \text{ and } |\varphi_{n_0}(x)| \text{ is bounded on G.}$$

(ii) Let $f(x)$ is bounded on G, i. e. $|f(x)|\leqq M$ on G. For any $\varepsilon>0$ and $\sigma>0$, there exists a continuous function $g(x)$ on G such that $|g(x)|\leqq M$ on G and

$$\mu(\{x;\ |f(x)-g(x)|\geqq\sigma\}<\varepsilon.$$

This statement is proved by the same method as the proof of Borel's theorem which is stated for G=[0, 1] (cf. I. P. Natanson[8]).

Namely, for such nutural number l as $M/l<\sigma$, putting

$$\mathrm{E}_i=\{x;\ (i-1)M/l\leqq f(x)\leqq iM/l \text{ and } x\in \mathrm{G}\}\quad (i=1-l, 2-l, \cdots, l-1)$$

and
$$\mathrm{E}_l=\{x;\ (l-1)M/l\leqq f(x)\leqq M\},$$

we get a partition $\{\mathrm{E}_i\}$ $(i=1-l, 2-l, \cdots, l)$ of G.

Since E_i are Lebesgue measurable sets, we can select closed sets F_i such that

$$\mu(\mathrm{F}_i)>\mu(\mathrm{E}_i)-\varepsilon/2l \quad\text{and}\quad \mathrm{F}_i\subset\mathrm{E}_i.$$

Defining a continuous function $g_1(x)$ on $\mathrm{F}=\bigcup_{i=1-l}^{l}\mathrm{F}_i$ such as

$$g_1(x)=iM/l \quad\text{if } x\in\mathrm{F}_i \quad (i=1-l, 2-l, \cdots, l),$$

we have $|f(x)-g_1(x)| \leqq M/l<\sigma$ for $x\in\mathrm{F}$.

Further, we get a continuous function $g(x)$ on G such that it is a extension of $g_1(x)$ on G, for which satisfies

$$|g(x)|\leqq M \quad\text{and } g(x)=0 \quad\text{if } x\in \mathrm{G}-\mathrm{F}.$$

The function $g(x)$ is the requirement.

(iii) By (ii), there exists a sequence $\{g_n(x)\}$ of continuous functions on G such that it converges in measure on G. Therefore we have, by Lebesgue's theorem,

$$\lim_{k\to\infty}\int_\Delta \Phi_2(|f(x)-g_{n_k}(x)|, x)\,dx=0$$

for some subsequence of $\{g_n(x)\}$. Accordingly, we have

$$\lim_{k\to\infty}\|f-g_{n_k}\|_\Phi=0,$$

and hence there exists a continuous function $g(x)$ on G such that

$$\|f-g\|_\Phi<\varepsilon.$$

(iv) If we assume that $f(x+h)=g(x+h)=0$ for $x+h\notin$ G, then we have, for enough small $\|h\|$, $\|g(x+h)-g(x)\|_\Phi<\varepsilon$ and $\|f(x+h)-g(x+h)\|_\Phi<\varepsilon$, which implies the required fact, i. e.

$$\|\varphi(x+h)-\varphi(x)\|_\Phi<5\varepsilon.$$

Remark. Suppose $\varDelta$ is a bounded set in $\mathbf{R}_n$. Let Φ_i, Ψ_i $(i=1, 2)$ be Young's functions satisfying the $(\varDelta_2)$-condition.

If
$$\int_\varDelta\int_\varDelta \Psi(R(s, t))\, ds\, dt < +\infty$$
where $\Psi \equiv \Phi_2[\Psi_1]$, then the linear operator $\int_\varDelta R(s, t)\phi(t)\, dt$ satisfies the conditions in Theorem 3, and hence the operator is a compact operator from $\mathbf{S}_1 \subset \mathbf{L}^*_{\Phi_1}$ into $\mathbf{L}^*_{\Phi_2}$. (cf. A. C. Zaanen[15], Krasnoselskii and Ya. B. Rutitskii ; Dokl. Akad. Nauk SSSR (n. s) 85 (1952), 33–36. Russian)
Because, by Lemma 2, we have
$$\lim_{\|n\|,\ \|k\|\to 0}\int_\varDelta\int_\varDelta \Psi(|R(s+h, t+k)-R(s, t)|)\, ds\, dt = 0$$
and hence
$$\lim_{\|h\|\to 0}\int_\varDelta \Psi_1(|R(s+h, t)-R(s, t)|\, dt = 0 \qquad \text{for almost all } s \in \varDelta.$$
And, we have also
$$\int_\varDelta |R(s+h, t)-R(s, t)|\cdot|\phi(t)|\, dt \leqq \|R(s+h, t)-R(s, t)\|_{\Psi_1} \qquad \text{for } \phi \in \mathbf{S}.$$
Namely, the assumptions of Theorem 3 are satisfied.

5. Combined the results in the section 3 with those in the section 4, we get the conditions of the complete continuity of the operator.

Theorem 4. *Let $\mathbf{L}_{\Phi_i}$ $(i=1, 2)$ are modulared function spaces with the bounded modulars, defining on a bounded set $\varDelta$ in $\mathbf{R}_n$. Let $K[s, t, u]$ be continuous in u $(-\infty < u < +\infty)$ for fixed (s, t) and measurable on $\varDelta\times\varDelta$ for fixed u satisfying the follywing conditions :*

a) *for every bounded measurable function $h(s, t)$ on $\varDelta\times\varDelta$*
$$m_2\left(\left|\int_\varDelta K[s, t, h(s, t)]\, dt\right|, s\right) < +\infty\, ;$$

b) *for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\phi-\psi\|_{\Phi_1} < \delta$ implies*
$$m_2\left(\left|\int_F \left\{K[s, t, \phi]-K[s, t, \psi]\right\} dt\right|, s\right) < \varepsilon$$
for $\mathrm{mes}(F) < \delta$ $(F \subset \varDelta)$;

c) $$\int_\varDelta \Big|K[x, t, \phi(t)]-K[s, t, \phi(t)]\Big|\, dt \leqq f(s)\, p(h) \quad (\phi \in \mathbf{S}_1)$$
for $\|x-s\| < h$, where $f(s) \in \mathbf{L}_{\Phi_2}$ and $p(h)$ tends to zero as $h \to 0$, then the operator $\mathbf{A}\phi(s)$ acts from $\mathbf{S}_1 \subset \mathbf{L}_{\Phi_1}$ into $\mathbf{L}_{\Phi_2}$ and is completely continuous.

Theorem 5. *In Theorem 4, if $\mathbf{L}_{\Phi_i}$ are Orlicz spaces, the condition* c) *is replaced by the following condition :*

c′) *for any bounded set $\mathfrak{A}$ in $\mathbf{L}_{\Phi_1}$ and almost all $s \in \varDelta$,*

$$\lim_{x \to s} \int_\Delta \Big| K[x, t, \phi(t)] - K[s, t, \phi(t)] \Big| dt = 0 \quad \textit{uniformly on } \mathfrak{A}$$

and

$$\sup_{\phi \in \mathfrak{A}} |\mathbf{A}\phi(s)| = f(s) \in \mathbf{L}_\Phi.$$

Remark. Under the assumptions in the *remark of the section* 3, we obtain that *the operator of Hammerstein type* $\mathbf{H}\phi(s)$ acts in Orlicz space $\mathbf{L}^*_\Phi$ and is completely continuous in the unit sphere $\mathbf{S}_1$ of $\mathbf{L}^*_\Phi$. Since it is shown that the operator acts in $\mathbf{L}^*_\Phi$ and is continuous, it is sufficient to show the compactness of the operator.

Putting

$$K[s, t, u] = R(s, t) f(t, u)$$

we have for any $\phi \in \mathbf{S}_1$

$$\int_\Delta \Big| K[x, t, \phi] - K[s, t, \phi] \Big| dt = \int_\Delta \Big| R(x, t) - R(s, t) \Big| \ \Big| f(t, \phi) \Big| dt$$
$$\leqq \| R(x, t) - R(s, t) \|_{\Psi_1} \ \| f(t, \phi) \|_{\Psi_1} \leqq \| R(x, t) - R(s, t) \|_{\Psi_1} \cdot M$$

and

$\| R(s, t) \|_{\Psi_1} \in \mathbf{L}^*_\Phi$ from the assumption 1), where $\sup_{\varphi \in \mathbf{S}_1} \| f(t, \phi) \|_{\Psi_1} = M < \infty$,

because the operator $\mathbf{f}: \ \mathbf{L}^*_\Phi \in \phi(t) \to f(t, \phi) \in \mathbf{L}^*_{\Phi_1}$ is bounded[12]. Therefore, on the assumptions 1), 2) and Lemma 2, we will know that the conditions ($\#$) and ($\#\#$) in Theorem 3 are satisfied, namely the operator $\mathbf{H}\phi(s)$ is compact.

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